



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function

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ARTICLE INFO

Article history:

Received 9 September 2008

Available online 5 February 2009

Submitted by B.C. Berndt

Keywords:

Gamma function

Double gamma function

Barnes G-function

Completely monotonic function of positive order

Asymptotic expansion

Strong complete monotonicity

ABSTRACT

We introduce completely monotonic functions of order $r > 0$ and show that the remainders in asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function give rise to completely monotonic functions of any positive integer order.

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1. Completely monotonic functions of positive order

In this paper it is established that the remainders in asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function are (up to a sign) completely monotonic functions of order comparable with the decay of the remainder in the expansion.

We first recall some definitions and give some preliminary results.

A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called completely monotonic if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } x > 0 \text{ and } n = 0, 1, 2, \dots \quad (1)$$

J. Dubourdieu [3] proved that if a nonconstant function f is completely monotonic then strict inequality holds in (1). See also [6] for a simpler proof of this result. A characterization of completely monotonic functions is given by Bernstein's theorem, see [14, p. 161], which states that f is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

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Here we are interested in the class of strongly completely monotonic functions, introduced in [13]. A function $f: (0, \infty) \rightarrow \mathbb{R}$ is called strongly completely monotonic if it has derivatives of all orders and $(-1)^n x^{n+1} f^{(n)}(x)$ is non-negative and decreasing on $(0, \infty)$ for all $n = 0, 1, 2, \dots$ (It is clear that being strongly completely monotonic is *stronger* than being completely monotonic.) These functions are connected to the important question of superadditivity (cf. [13]).

The following proposition contains a simple characterization of strongly completely monotonic functions and its proof is obtained by a direct application of the definitions given above.

Proposition 1.1. *A function $f(x)$ is strongly completely monotonic if and only if the function $xf(x)$ is completely monotonic.*

In [13] the authors gave another characterization of strongly completely monotonic functions.

Proposition 1.2. *The function $f(x)$ is strongly completely monotonic if and only if*

$$f(x) = \int_0^{\infty} e^{-xt} p(t) dt,$$

where $p(t)$ is nonnegative and increasing and the integral converges for all $x > 0$.

We notice that any right-continuous function p appearing in Proposition 1.2 can be written as

$$p(t) = \mu([0, t]),$$

where μ is a Radon measure on $[0, \infty)$.

An extension of Propositions 1.1 and 1.2 is the following.

Theorem 1.3. *Let r be an integer ≥ 1 . The function $x^r f(x)$ is completely monotonic if and only if*

$$f(x) = \int_0^{\infty} e^{-xt} p(t) dt,$$

where the integral converges for all $x > 0$ and where p is $r - 1$ times differentiable on $[0, \infty)$ with $p^{(r-1)}(t) = \mu([0, t])$ for some Radon measure μ and $p^{(k)}(0) = 0$ for $0 \leq k \leq r - 2$.

Remark 1.4. The conditions on p ensure that $p, p', \dots, p^{(r-1)}$ are all nonnegative.

Proof. Suppose that $x^r f(x)$ is completely monotonic. Then

$$x^r f(x) = \int_0^{\infty} e^{-xt} d\mu(t) = \mathcal{L}(\mu)(x),$$

for some Radon measure μ (and where \mathcal{L} denotes the Laplace transform). Furthermore, it is a fact that

$$x^{-r} = \frac{1}{(r-1)!} \int_0^{\infty} t^{r-1} e^{-xt} dt,$$

and this yields $f(x) = \mathcal{L}(p)(x)$ where p is the convolution of these two measures on the half line,

$$p(t) = \left(\frac{s^{r-1}}{(r-1)!} ds * \mu \right)(t) = \frac{1}{(r-1)!} \int_0^t (t-s)^{r-1} d\mu(s).$$

From this formula it is easy to check that p has the asserted properties.

Conversely, if

$$p^{(r-1)}(t) = \mu([0, t])$$

and all derivatives of p at $t = 0$ up to order $r - 2$ are zero then we find by integration ($r - 1$ times)

$$p(t) = \frac{1}{(r-1)!} \int_0^t (t-s)^{r-1} d\mu(s).$$

This gives

$$x^r f(x) = x^r \mathcal{L}(p)(x) = x^r \mathcal{L}\left(\frac{s^{r-1}}{(r-1)!} ds * \mu\right)(x) = x^r \mathcal{L}\left(\frac{s^{r-1}}{(r-1)!} ds\right)(x) \mathcal{L}(\mu)(x) = \mathcal{L}(\mu)(x),$$

and therefore $x^r f(x)$ is completely monotonic. \square

In the light of these results we formulate the following definition.

Definition 1.5. Let $r \geq 0$. A function f defined on $(0, \infty)$ is said to be completely monotonic of order r if $x^r f(x)$ is completely monotonic.

According to this definition, completely monotonic functions of order 0 are the classical completely monotonic functions, order 1 are the strongly completely monotonic functions and so on.

Theorem 1.3 is the basis of the proofs of Theorems 2.1, 3.1 and 4.1.

Remark 1.6. Following [1, Section 2.9] a fractional integral $I_\alpha(\mu)(t)$ (for $\alpha > 0$) of a measure on $[0, \infty)$ is defined by

$$I_\alpha(\mu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\mu(s).$$

The proof of Theorem 1.3 actually shows that f is completely monotonic of order $\alpha > 0$ if and only if f is the Laplace transform of a fractional integral of a positive Radon measure on $[0, \infty)$, that is,

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xt} \int_0^t (t-s)^{\alpha-1} d\mu(s) dt.$$

2. Euler's gamma function

Let us consider the asymptotic expansion of the logarithm of Euler's gamma function

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + \sum_{k=1}^n \frac{B_{2k}}{(2k-1)2k} \frac{1}{x^{2k-1}} + (-1)^n R_n(x), \quad (2)$$

where B_{2k} are the Bernoulli numbers. It has been shown in [2] that for all $n = 0, 1, 2, \dots$, the remainder $R_n(x)$ is completely monotonic on $(0, \infty)$. We strengthen this result in Theorem 2.1 below.

For $t > 0$ we write

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^n \frac{B_{2k}}{(2k)!} t^{2k} + (-1)^n t^{2n+2} V_n(t), \quad (3)$$

where the remainder term $V_n(t)$ is given as (see [12, p. 64])

$$V_n(t) = \sum_{k=1}^{\infty} \frac{2}{(t^2 + 4\pi^2 k^2)(2\pi k)^{2n}}, \quad n \geq 0. \quad (4)$$

Theorem 2.1. The function $R_n(x)$ defined in (2) is completely monotonic of order k for $n \geq k$ for any $k \geq 0$. Indeed,

$$R_n(x) = \int_0^\infty e^{-xt} r_n(t) dt,$$

where $r_n(t) = t^{2n} V_n(t)$ satisfies $r_n^{(l)}(0) = 0$ for $0 \leq l \leq k-1$ and $r_n^{(k)}(t) > 0$ for $t > 0$.

A similar result regarding complete monotonicity of the remainder of an asymptotic expansion of a ratio of gamma functions of the form $\Gamma(x+a)/\Gamma(x+b)$ in terms of powers of $1/(x+c)$, is given in Frenzen's paper [5].

Remark 2.2. In [7] it is proved that $V_n(t)$ takes also the form

$$V_n(t) = \frac{1}{(2n+1)!} \frac{1}{e^t - 1} \int_0^1 e^{tu} (-1)^n B_{2n+1}(u) du, \quad n \geq 0,$$

where $B_{2n+1}(u)$ are the Bernoulli polynomials. We observe that this formula extends to $t = 0$ because of the well-known property

$$\int_0^1 B_{2n+1}(u) du = 0$$

for $n \geq 0$.

In [8] it is proved that V_n is positive, decreasing and satisfies $(t^2 V_n(t))' > 0$. We extend the last property in the following Lemma 2.3.

Lemma 2.3. *The function V_n has the following properties:*

- (i) $(t^{2j} V_n(t))^{(l)} > 0$ for all $n \geq 0$ and all $l \leq j$,
- (ii) $(t^{2j-1} V_n(t))^{(l)} > 0$ for all $n \geq 0$ and all $l \leq j - 1$.

Proof. We have

$$t^{2j} V_n(t) = 2 \sum_{k=1}^{\infty} \frac{t^{2j}}{t^2 + (2\pi k)^2} \frac{1}{(2\pi k)^{2n}} = 2 \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^{2n-2j+2}} s_j(t/(2\pi k)),$$

where

$$s_j(x) = \frac{x^{2j}}{1+x^2}.$$

Here we notice that $s_j^{(l)}(x)$ tends to zero as x^{2j-l} for x tending to 0. We get, by differentiation,

$$(t^{2j} V_n(t))^{(l)} = 2 \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^{2n-2j+2}} s_j^{(l)}(t/(2\pi k)) \frac{1}{(2\pi k)^l},$$

where the series converges uniformly due to the behaviour of $s_j^{(l)}(t/(2\pi k))$ for large k . We have $s_j(x) = \xi_j(x^2)$, where

$$\xi_j(x) = \frac{x^j}{1+x}$$

and from Lemma 2.4 below it follows that $s_j^{(l)}(x)$ are all positive. This proves the first assertion. The second assertion is proved in the same way. (Of course the first assertion follows from the second when $l \leq j - 1$.) \square

Lemma 2.4. *Let*

$$\xi_n(x) = \frac{x^n}{1+x}.$$

Then $\xi_n^{(k)}(x) > 0$ for $k \leq n$ and $x > 0$.

Proof. First of all we notice that $\xi_n^{(k)}(0) = 0$ for $k = 0, \dots, n - 1$. To consider the n th derivative, $\xi_n(x)$ is rewritten as follows:

$$\xi_n(x) = \frac{x^n}{1+x} = \frac{1}{x+1} \sum_{k=0}^n \binom{n}{k} (x+1)^k (-1)^{n-k} = \frac{(-1)^n}{x+1} + \sum_{k=1}^n \binom{n}{k} (x+1)^{k-1} (-1)^{n-k}$$

so that

$$\xi_n^{(n)}(x) = (-1)^n (x+1)^{-(n+1)} (-1)^n n! = \frac{n!}{(x+1)^{n+1}} > 0.$$

Using the relation

$$\xi_n^{(k)}(x) = \int_0^x \xi_n^{(k+1)}(t) dt$$

recursively for k from $n - 1$ to 0, we find that the derivatives of ξ_n of order not exceeding n are positive. \square

In the next proposition we gather some additional properties of the function V_n that are of interest in their own right.

Proposition 2.5. *The function V_n has the following additional properties.*

- (i) $t^{2n}V_n(t) = (-1)^n V_0(t) + \sum_{k=1}^n (-1)^{n-k} V_{n-k}(0)t^{2k-2}$, $n \geq 0$.
- (ii) $V_n(0) = (-1)^n \frac{B_{2n+2}}{(2n+2)!}$, $n \geq 0$.
- (iii) $V_n''(0) = -2V_{n+1}(0)$, $n \geq 0$.
- (iv) $V_n''(t) - V_n''(0) > 0$, $\forall t > 0$, $n \geq 0$.

Remark 2.6. The function V_0 appearing in this proposition satisfies by definition $t/(e^t - 1) = 1 - t/2 + t^2V_0(t)$, whence $V_0(t) = 1/t^2((t/2) \coth(t/2) - 1)$.

Proof. The first assertion is obtained by repeated application of the recursive relation

$$t^2V_n(t) = V_{n-1}(0) - V_{n-1}(t),$$

see [8]. The remaining assertions are easily obtained by using definition (3) and relation (4). \square

Proof of Theorem 2.1. Using Binet's formula

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + \int_0^\infty \left(\frac{t}{2} - 1 + \frac{t}{e^t - 1}\right) \frac{e^{-xt}}{t^2} dt, \quad x > 0,$$

and formula (3) we see that

$$r_n(t) = t^{2n}V_n(t), \quad n = 0, 1, 2, \dots$$

It is clear that $r_n^{(k)}(0) = 0$ for $k \leq n-1$. Now use (i) of Lemma 2.3 to prove that $r_n^{(n)}(t) > 0$ for $t > 0$. The result then follows from Theorem 1.3. \square

Corollary 2.7. *For any $n \geq 1$ the function*

$$F_n(x) = (-1)^n \left[x^n \log \Gamma(x) - x^n \left(x - \frac{1}{2}\right) \log x + x^{n+1} - \frac{x^n}{2} \log(2\pi) - \sum_{k=1}^n \frac{B_{2k}}{(2k-1)2k} x^{n-2k+1} \right]$$

is completely monotonic on $(0, \infty)$.

Proof. It follows by a combination of Theorem 2.1 with Theorem 1.3 and (2). \square

3. Barnes G-function

We note that a result similar to Theorem 2.1 holds for the remainder in an asymptotic expansion (due to C. Ferreira and J.L. López [4, Theorem 1]) of the logarithm of Barnes G-function. This function is defined as an infinite product and satisfies $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$. See also [9] for details and additional considerations. The remainder in this expansion takes the form

$$P_n(x) = (-1)^n \int_0^\infty e^{-xt} t^{2n-1} V_n(t) dt,$$

where $V_n(t)$ is as above.

Theorem 3.1. *The remainder $(-1)^n P_n(x)$ is completely monotonic of order k on $(0, \infty)$ for $n \geq k+1$.*

Proof. Let $\lambda_n(t) = t^{2n-1}V_n(t)$. Clearly, $\lambda_n^{(k)}(0) = 0$ for $k \leq n-1$. It follows from Lemma 2.3 that $\lambda_n^{(n-1)}(t) > 0$ for $t > 0$, and the result follows from Theorem 1.3. \square

4. Barnes double gamma function

This section is devoted to the investigation of the remainders in an asymptotic expansion due to Ruijsenaars of the logarithm of Barnes double gamma function. The expansion is given in terms of generalized Bernoulli polynomials $B_k^{(2)}(x)$, see [1, p. 615] or [12, p. 4]. Our investigation is based on Ruijsenaars' results and therefore we have found it natural to his

terminology $B_{2,k}(x) = B_k^{(2)}(x)$. We shall furthermore call these polynomials the double Bernoulli polynomials. (Ruijsenaars calls $B_{N,k}(x)$ multiple Bernoulli polynomials.)

The double Bernoulli polynomials $B_{2,k}(x)$ are defined by

$$\frac{t^2 e^{xt}}{(e^t - 1)^2} = \sum_{k=0}^{\infty} B_{2,k}(x) \frac{t^k}{k!}$$

and the double Bernoulli numbers $B_{2,k}$ by $B_{2,k} = B_{2,k}(0)$.

The asymptotic expansion of the logarithm of Barnes double gamma function with both parameters equal to 1, $\log \Gamma_2(w) = \log \Gamma_2(w|1, 1)$, is given as follows:

$$\log \Gamma_2(w) = -\frac{B_{2,2}(w)}{2} \log w + \frac{3}{4} B_{2,0} w^2 + B_{2,1} w + \sum_{k=3}^M \frac{(-1)^k}{k!} (k-3)! B_{2,k} w^{2-k} + R_{2,M}(w),$$

where the remainder $R_{2,M}$ has the representation

$$R_{2,M}(w) = \int_0^{\infty} \frac{e^{-wt}}{t^3} \left(\frac{t^2}{(1 - e^{-t})^2} - \sum_{k=0}^M \frac{(-1)^k}{k!} B_{2,k} t^k \right) dt.$$

Here, $\Re w > 0$ and $M \geq 2$. See [11, (3.13) and (3.14)].

In [8,10] it was shown independently that $(-1)^{n-1} R_{2,2n}(x)$ is a completely monotonic function. Below it is verified that it is indeed a completely monotonic function of order k for $n \geq k+1$.

We briefly indicate the two different proofs of complete monotonicity. By Bernstein's theorem it amounts to showing the positivity of $U_n(t)$ for $t > 0$ and any $n \geq 1$, where

$$U_n(t) = (-1)^{n-1} \left(\frac{t^2}{(1 - e^{-t})^2} - \sum_{k=0}^{2n} \frac{(-1)^k}{k!} B_{2,k} t^k \right),$$

since

$$(-1)^{n-1} R_{2,2n}(x) = \int_0^{\infty} e^{-xt} \frac{U_n(t)}{t^3} dt. \quad (5)$$

In [8] it was shown that

$$U_n(t) = t^{2n+1} V_{n-1}(t) + t^2 (t^{2n+1} V_n(t))', \quad (6)$$

where $V_n(t)$ is defined in (3) and the proof is obtained by showing that $(t^{2n+1} V_n(t))' > 0$. In [10] the proof is based on a contour integration argument and the following representation of $p_n(t) = U_n(t)/t^3$ in (5) is found

$$p_n(t) = t^{2n-2} \sum_{k=1}^{\infty} (2\pi k)^{1-2n} \left(\frac{4\pi k}{t^2 + (2\pi k)^2} + \frac{8\pi k t}{(t^2 + (2\pi k)^2)^2} + \frac{(2n-1)}{2\pi k} \frac{2t}{t^2 + (2\pi k)^2} \right). \quad (7)$$

This clearly shows the positivity of U_n .

The main result is formulated in the theorem below.

Theorem 4.1. *The remainder $(-1)^{n-1} R_{2,2n}(x)$ is completely monotonic of order k on $(0, \infty)$ for $n \geq k+1$.*

Proof. It follows from (6) that

$$\begin{aligned} p_n(t) &= \frac{U_n(t)}{t^3} = t^{2n-2} V_{n-1}(t) + (2n+1) t^{2n-1} V_n(t) + t^{2n} V'_n(t) = t^{2n-2} V_{n-1}(t) + t^{2n-1} V_n(t) + (t^{2n} V_n(t))' \\ &= r_{n-1}(t) + \lambda_n(t) + r'_n(t), \end{aligned}$$

where $r_n(t) = t^{2n} V_n(t)$ and $\lambda_n(t) = t^{2n-1} V_n(t)$.

Clearly $p_n^{(k)}(0) = 0$ for $k \leq n-2$. Since $r_n^{(k)}(t) > 0$ for $0 \leq k \leq n$ and $\lambda_n^{(k)}(t) > 0$ for $0 \leq k \leq n-1$, $t > 0$, it follows from the above that $p_n^{(k)}(t) > 0$ for $0 \leq k \leq n-1$ and for all $t > 0$ and by Theorem 1.3 this completes the proof of the theorem. \square

Remark 4.2. Theorem 4.1 can also be obtained directly from the representation (7). Indeed, since

$$t^{2n-2} \sum_{k=1}^{\infty} (2\pi k)^{1-2n} \frac{4\pi k}{t^2 + (2\pi k)^2} = \sum_{k=1}^{\infty} \frac{2}{(2\pi k)^2} \frac{(t/(2\pi k))^{2n-2}}{1 + (t/(2\pi k))^2},$$

$$t^{2n-2} \sum_{k=1}^{\infty} (2\pi k)^{1-2n} \frac{8\pi k t}{(t^2 + (2\pi k)^2)^2} = \sum_{k=1}^{\infty} \frac{4t}{(2\pi k)^4} \frac{(t/(2\pi k))^{2n-2}}{(1 + (t/(2\pi k))^2)^2}$$

and

$$t^{2n-2} \sum_{k=1}^{\infty} (2\pi k)^{1-2n} \frac{(2n-1)}{2\pi k} \frac{2t}{t^2 + (2\pi k)^2} = \sum_{k=1}^{\infty} \frac{2(2n-1)t}{(2\pi k)^4} \frac{(t/(2\pi k))^{2n-2}}{1 + (t/(2\pi k))^2}$$

we have from (7)

$$p_n(t) = \sum_{k=1}^{\infty} \frac{2}{(2\pi k)^2} s_{n-1}(t/(2\pi k)) + \sum_{k=1}^{\infty} \frac{4t}{(2\pi k)^4} g_{n-1}(t/(2\pi k)),$$

where g_n is defined by

$$g_n(x) = (n+1/2) \frac{x^{2n}}{1+x^2} + \frac{x^{2n}}{(1+x^2)^2}.$$

We put

$$h_n(x) = n \frac{x^n}{1+x} + \frac{x^n}{(1+x)^2}$$

and have in this way $g_n(x) = s_n(x)/2 + h_n(x^2)$. The positivity of the n th derivative of g_n clearly follows from the positivity of the derivatives $h_n^{(k)}$ for $k \leq n$. To investigate the derivatives of h_n we rewrite it as follows.

$$\begin{aligned} h_n(x) &= \frac{nx^n}{x+1} + \frac{x^n}{(x+1)^2} \\ &= \frac{n}{x+1} \left\{ (-1)^n + \sum_{k=1}^n \binom{n}{k} (x+1)^k (-1)^{n-k} \right\} + \frac{1}{(x+1)^2} \left\{ (-1)^n + n(x+1)(-1)^{n-1} + \sum_{k=2}^n \binom{n}{k} (x+1)^k (-1)^{n-k} \right\} \\ &= \frac{(-1)^n}{(x+1)^2} + l_n(x), \end{aligned}$$

where l_n is a polynomial of degree $n-1$. Therefore

$$h_n^{(n)}(x) = \frac{(n+1)!}{(1+x)^{n+2}} > 0.$$

Furthermore, $h_n^{(k)}(0) = 0$ for $k \leq n-1$, whence

$$h_n^{(k-1)}(x) = \int_0^x h_n^{(k)}(t) dt > 0$$

for $k = 1, \dots, n$. This completes a different proof of Theorem 4.1.

Acknowledgment

The authors thank Christian Berg for his comments in particular regarding Theorem 1.3.

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